Accelerating the Delfs–Galbraith Algorithm with Fast Subfield Root Detection

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Based on joint work with Craig Costello and Jia Shi

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Outline

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The Supersingular Isogeny Problem

In its most general form, the *supersingular isogeny problem* asks to find an isogeny

$$\phi : E_1 \to E_2,$$

between two given supersingular elliptic curves $E_1/\mathbb{F}_{p^2}$ and $E_2/\mathbb{F}_{p^2}$.

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The best known classical attack against this general problem is the **Delfs–Galbraith algorithm**.

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- Use this to introduce an improved attack, SuperSolver, with lower concrete complexity.

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- Ramanujan graph: *rapid mixing*.

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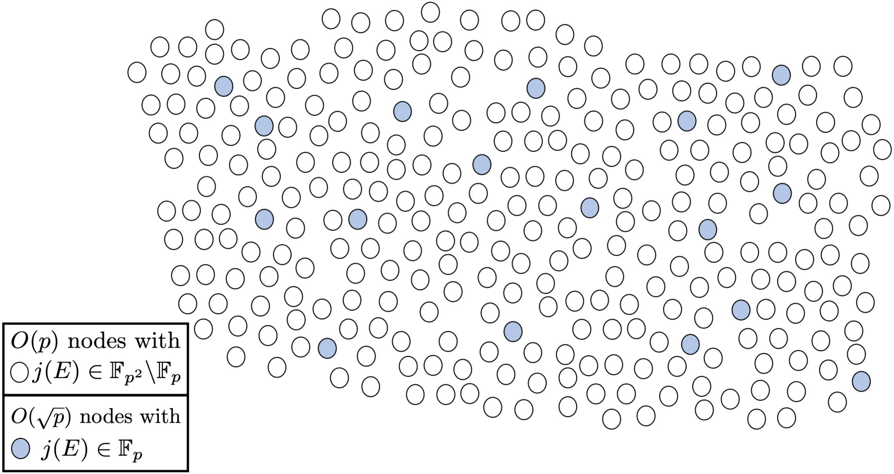
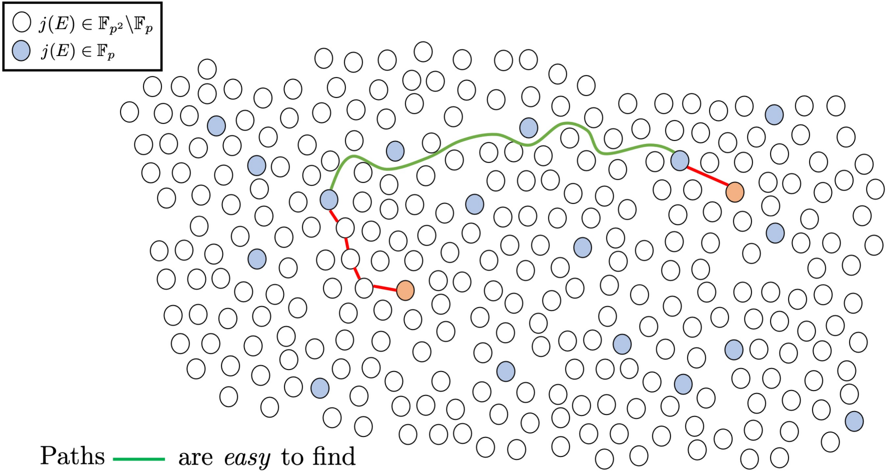


Outline

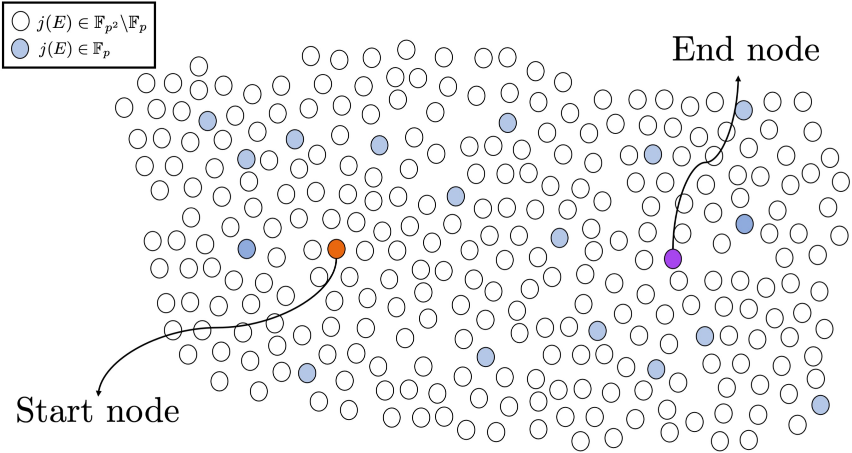
Key Observation



Paths —— are *easy* to find

Finding paths —— is the bottleneck

The Delfs–Galbraith Algorithm



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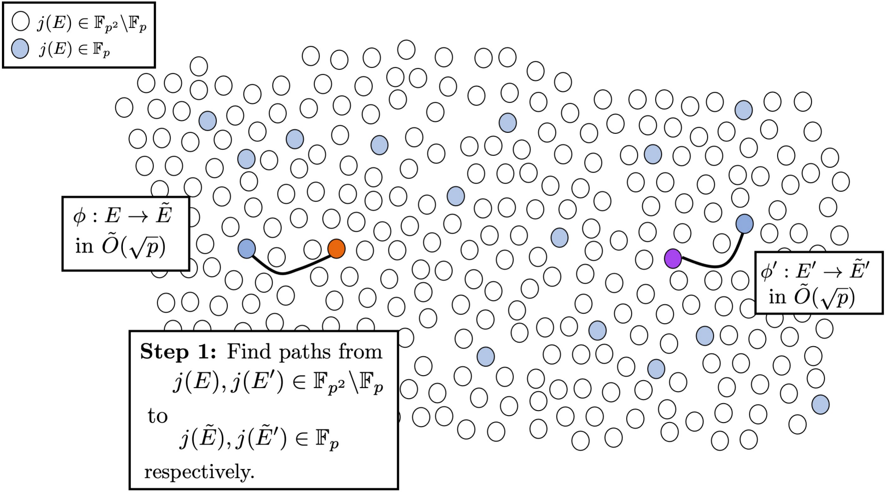

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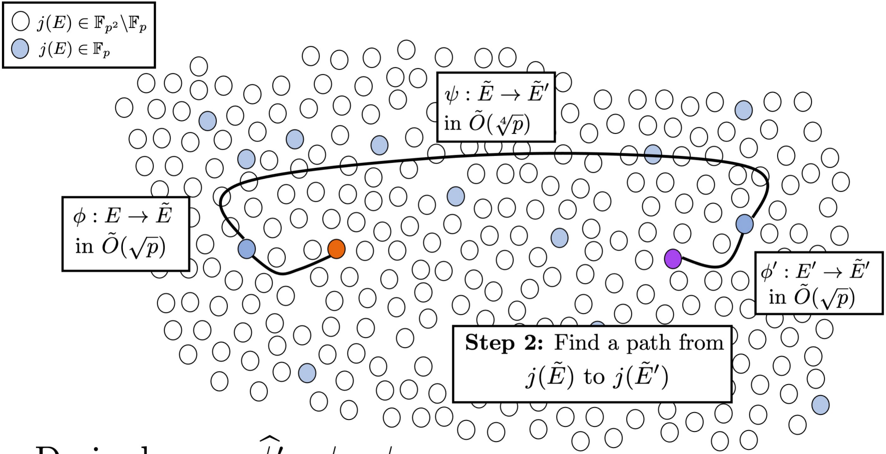

Desired map: $\widehat{\phi'} \circ \psi \circ \phi$

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- symmetric in X and Y
- of degree N_ℓ in both X and Y, where

$$N_\ell := \prod_{i=1}^{n} (\ell_i + 1)\ell_i^{e_i - 1}, \text{ for prime decomposition } \prod_{i=1}^{n} \ell_i^{e_i} \text{ of } \ell.$$

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This tells us that the roots of $\Phi_{\ell,p}(X, j)$ are neighbours of j in $\mathcal{X}(\mathbb{F}_p, \ell)$. Reducing coefficients $\bmod p$ we can work with $\Phi_{\ell,p}(X, Y) \in \mathbb{F}_p[X, Y]$.

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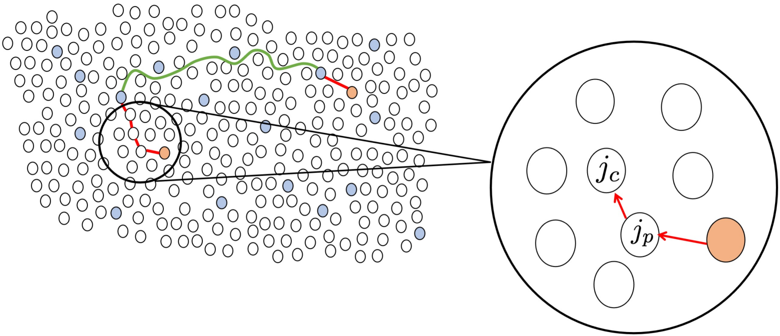

1. Store the current and previous j-invariants j_c and j_p.

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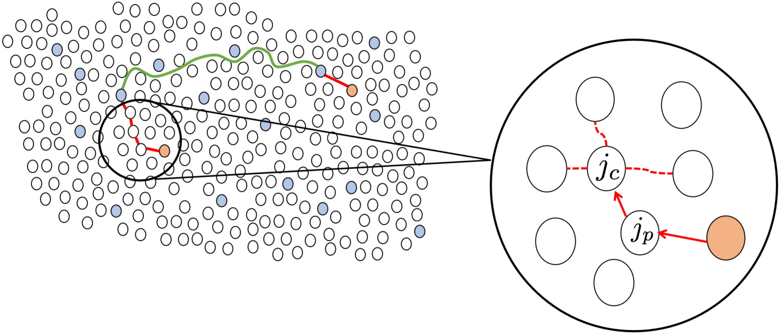

2. Find the $N_\ell - 1$ roots of $\Phi_{\ell,p}(X, j_c)/(X - j_p)$.

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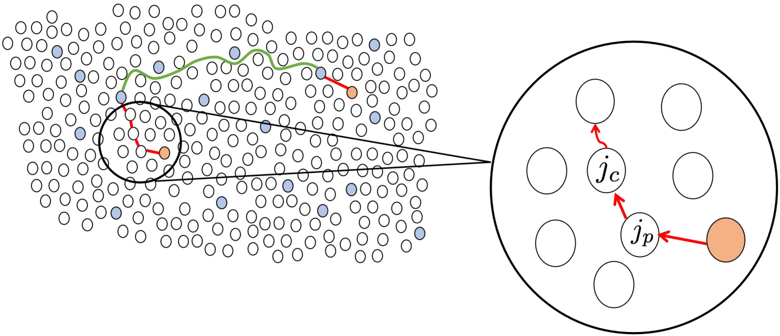

3. Choose one of these and walk to the corresponding node.

Concrete Complexity of Delfs–Galbraith

Solver is an optimised implementation of the Delfs–Galbraith algorithm with $\ell = 2$.

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Experimentally, given a node $j \in \mathbb{F}_{p^2} \backslash \mathbb{F}_p$, the average number of $\mathbb{F}_p$ multiplications needed to find a path to a node $j' \in \mathbb{F}_p$ is

$$c \cdot \sqrt{p} \cdot \log_2 p,$$

with $0.75 \leq c \leq 1.05$.

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Key Observation

At each step, the precise values of the ℓ-isogenous neighbours do not need to be known, only whether it lies in $\mathbb{F}_p$.

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At each step of the random walk in $\mathcal{X}(\bar{\mathbb{F}}_p, 2)$, SuperSolver inspects the ℓ-isogeny graph with fast subfield root detection for ℓ in a carefully chosen set, to efficiently detect whether j_c has an ℓ-isogenous neighbour in $\mathbb{F}_p$.

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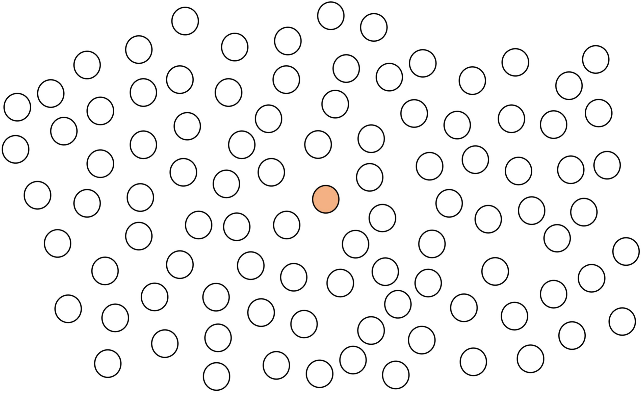

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Lemma

Let π be the p-power Frobenius map and f a polynomial in $\mathbb{F}_{p^2}[X]$. Then, $\gcd(f, \pi(f))$ is the largest divisor of f defined over $\mathbb{F}_p$.
In particular, if

$$\deg\left(\gcd(f, \pi(f))\right) = \begin{cases} 1, & f \text{ has a root in } \mathbb{F}_p \\ 0, & f \text{ does not have a root in } \mathbb{F}_p \end{cases}.$$

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Observation

For polynomials $f_1, f_2 \in \mathbb{F}_{p^2}[X]$, if

$$g_1 = af_1 + bf_2, \text{ and } g_2 = cf_1 + df_2,$$

with $a, b, c, d \in \mathbb{F}_{p^2}$ such that $ad - bc \neq 0$ with we have

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We can avoid **all** multiplications over $\mathbb{F}_{p^2}$: if we write the coefficients of $f(X)$ as $a_k^{(1)} + a_k^{(2)}\alpha$ (say $\alpha^2 = -1$), then

$$g_1(X) = \sum_{k=0}^{n} a_k^{(1)} X^k, \text{ and } g_2(X) = \sum_{k=0}^{n} a_k^{(2)} X^k.$$

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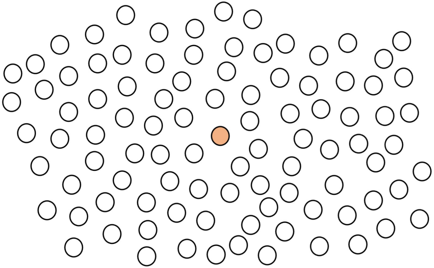

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Calculating the list of optimal ℓ's can be done in precomputation.

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Worked Example: Precomputation

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- Construct the extension field $\mathbb{F}_{p^2} = \mathbb{F}_p(\alpha)$, where α^2 is the first non-square in $-1, -2, 2, -3, 3, \dots$.

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- For SuperSolver, compute a list of optimal ℓ's L.

Worked Example: Precomputation

Let $p = 2^{20} - 3$.

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- Reduces the coefficients of $\Phi_\ell(X, Y) \in \mathbb{Z}[X, Y]$ mod p to obtain $\Phi_{\ell,p}(X, Y) \in \mathbb{F}_p[X, Y]$.
- For SuperSolver, compute a list of optimal ℓ's L.

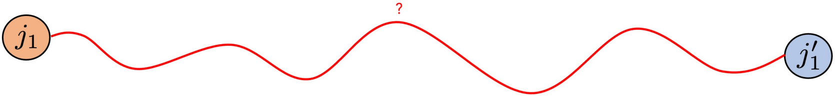
Sample our start and end node:

Start Node: $j_1 = 129007\alpha + 818380$

End Node: $j_2 = 97589\alpha + 660383$

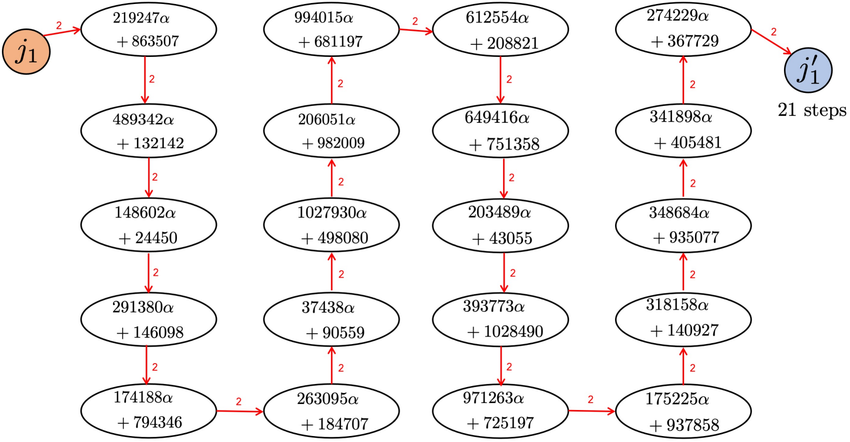
Worked Example: Solver

Path from $j_1 = 129007\alpha + 818380$ to subfield node.



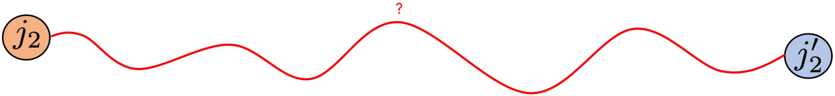
Worked Example: Solver

Path from $j_1 = 129007\alpha + 818380$ to subfield node $j_1' = 760776$.



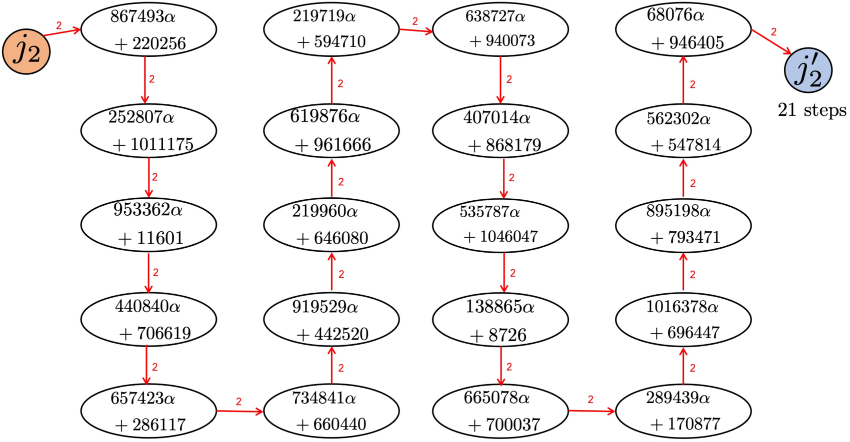
Worked Example: Solver

Path from $j_2 = 97589\alpha + 660383$ to subfield node.



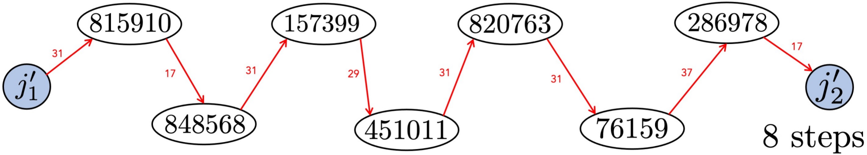
Worked Example: Solver

Path from $j_2 = 97589\alpha + 660383$ to subfield node $j_2' = 35387$.



Worked Example: Solver

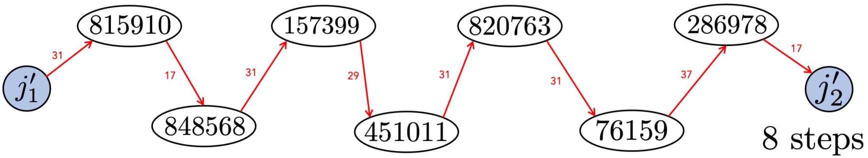
Path between subfield nodes $j'_1 = 760776$ and $j'_2 = 35387$.



We take steps in $\mathcal{X}(\bar{\mathbb{F}}_p, \ell)$ with $\ell \in \{17, 29, 31, 37\}$.

Worked Example: Solver

Path between subfield nodes $j_1' = 760776$ and $j_2' = 35387$.



We take steps in $\mathcal{X}(\bar{\mathbb{F}}_p, \ell)$ with $\ell \in \{17, 29, 31, 37\}$.

In total, the path has $21 + 21 + 8 =$ **50 steps**.

Worked Example: SuperSolver

The list of optimal ℓ's is precomputed as $L = \{3, 5\}$.

Worked Example: SuperSolver

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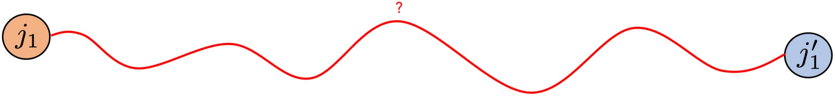
Path from $j_1 = 129007\alpha + 818380$ to subfield node.



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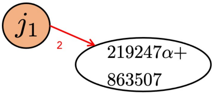

3-isogenous neighbour in $\mathbb{F}_p$?

$$\Phi_{3,p}(X, 219247\alpha + 863507) = X^4 + (212814\alpha + 479338)X^3 + (408250\alpha + 920025)X^2 + (811739\alpha + 93038)X + 942336\alpha + 847782$$

Worked Example: SuperSolver

The list of optimal ℓ's is precomputed as $L = \{3, 5\}$.
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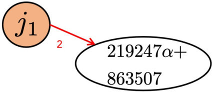

3-isogenous neighbour in $\mathbb{F}_p$?

$$g_1 = X^4 + 479338X^3 + 920025X^2 + 93038X + 847782$$
$$g_2 = 425628X^3 + 816500X^2 + 574905X + 836099$$

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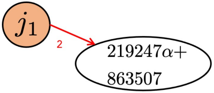

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$$\boxed{\gcd(g_1, g_2) = 1 \implies \text{no 3-isogenous neighbour in } \mathbb{F}_p}$$

Worked Example: SuperSolver

The list of optimal ℓ's is precomputed as $L = \{3, 5\}$.
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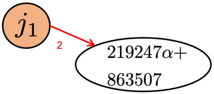

3-isogenous neighbour in $\mathbb{F}_p$? No.
Similarly, no 5-isogenous neighbour in $\mathbb{F}_p$.

Worked Example: SuperSolver

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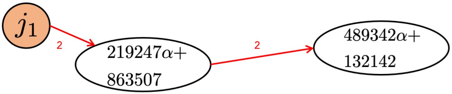
Path from $j_1 = 129007\alpha + 818380$ to subfield node $j_1' = 35387$.



3-isogenous neighbour in $\mathbb{F}_p$?

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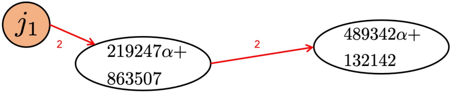
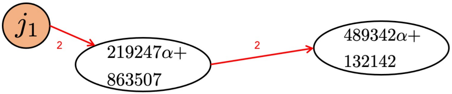


3-isogenous neighbour in $\mathbb{F}_p$?

$$\Phi_{3,p}(X, 489342\alpha + 132142) = X^4 + (872004\alpha + 13960)X^3 + (1031755\alpha + 822066)X^2 + (969683\alpha + 747785)X + 813010\alpha + 255391.$$

Worked Example: SuperSolver

The list of optimal ℓ's is precomputed as $L = \{3, 5\}$.
Path from $j_1 = 129007\alpha + 818380$ to subfield node $j_1' = 35387$.



3-isogenous neighbour in $\mathbb{F}_p$?

$$g_1 = X^4 + 13960X^3 + 822066X^2 + 747785X + 255391$$
$$g_2 = 695435X^3 + 1014937X^2 + 890793X + 577447$$

$$\gcd(g_1, g_2) = X + 1013186 \implies \text{3-isogenous neighbour in } \mathbb{F}_p$$
$$-1013186 = 35387$$

Worked Example: SuperSolver

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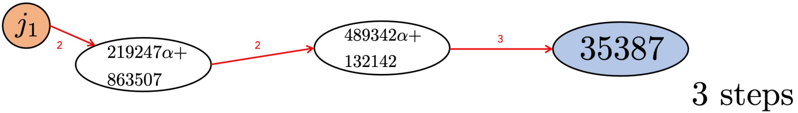
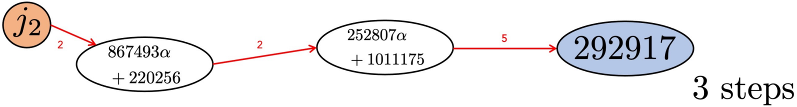


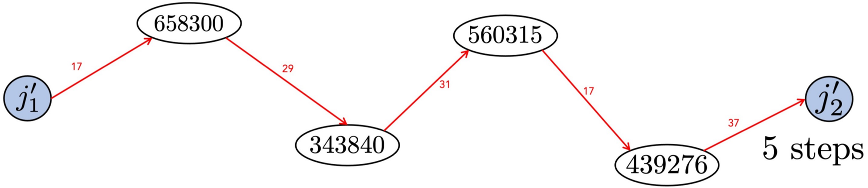
Worked Example: SuperSolver

The list of optimal ℓ's is precomputed as $L = \{3, 5\}$.
Path from $j_2 = 97589\alpha + 660383$ to subfield node $j_2' = 292917$.



Worked Example: SuperSolver

The list of optimal ℓ's is precomputed as $L = \{3, 5\}$.
Path between subfield nodes $j_1' = 35387$ and $j_2' = 292917$.



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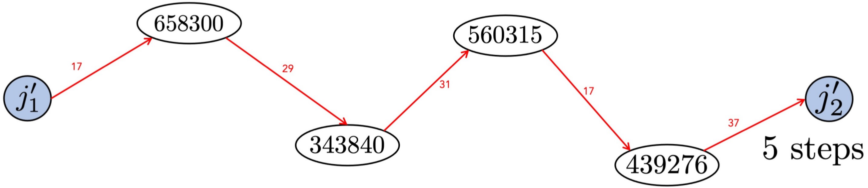

We take steps in $\mathcal{X}(\bar{\mathbb{F}}_p, \ell)$ with $\ell \in \{17, 29, 31, 37\}$.

In total, the path has $3 + 3 + 5 =$ **11 steps**.

Outline

Results

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Example: For $p = 2^{24} - 3$, averaging over 5000 pseudo-random supersingular *j*-invarants in $\mathbb{F}_{p^2}$, we get:

Solver used *112878* $\mathbb{F}_p$ multiplications and walked on *1897* nodes.

SuperSolver used *53900* $\mathbb{F}_p$ multiplications and walked on *318* nodes.

Results

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Experiments on cryptographic sized primes and one j-invariant. We ran SuperSolver and Solver until the number of $\mathbb{F}_p$ multiplications used exceeded 10^8, recording the total number of nodes covered.

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Examples:

For $p = 2^{50} - 27$, SuperSolver covers between *3 and 4 times* the number of nodes that Solver does.

For $p = 2^{800} - 105$, SuperSolver covers between *18 and 19 times* the number of nodes.

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- We improve the concrete complexity of Delfs–Galbraith - asymptotic complexity is unchanged.
- No direct impact on SIDH and SIKE - there are faster claw-finding algorithms.
- Affects other proposals, such as B-SIDH and SQISign, with Delfs–Galbraith as their best attack.

Open Problems

Relating to SuperSolver:

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- Can we combine $\Phi_m(X, j)$ and $\Phi_n(X, j)$ so that we can detect whether j has an nm-isogenous neighbour doing operations with Φ_m and Φ_n only?

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- Can we combine $\Phi_m(X, j)$ and $\Phi_n(X, j)$ so that we can detect whether j has an *nm*-isogenous neighbour doing operations with Φ_m and Φ_n only?
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Relating to SuperSolver:

- Can we combine $\Phi_m(X, j)$ and $\Phi_n(X, j)$ so that we can detect whether j has an nm-isogenous neighbour doing operations with Φ_m and Φ_n only?
- What does a *quantum version* of SuperSolver look like?
- Other applications of subfield detection